# A Rigorous Technique for Measuring the Scattering Matrix of a Multiport Device with a 2-Port Network Analyzer

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*Abstract*—**A new measurement technique is described that eliminates the mismatched-induced errors that occur when the scattering matrix of a multiport device is measured with a 2-port network analyzer. These errors arise from neglecting the finite reflections from the imperfect auxiliary loads terminating the unused ports of the device under test in each of the required 2-port measurements. It is shown how a systematic application of the generalized scattering matrix renormalization transform completely eliminates these errors. This new method is completely general and can therefore be applied to measurements of the scattering matrix of an $n$-port device with an $m$-port network analyzer ($m < n$).**

## I. Introduction

ALTHOUGH a new method for determining the $n \times n$ complex scattering matrix of a multiport device in a single measurement was recently developed [1], this method requires a nonconventional type of multiport network analyzer. Alternatively, a more conventional way of performing such measurements on an existing 2-port automated network analyzer (ANA) is possible. Using the conventional method, however, several partial 2-port measurements must be performed with the ANA connected to various 2-port combinations of the device under test (DUT). In each of these 2-port measurements, the $n-2$ unused ports of the DUT should ideally be terminated with perfectly matched loads. Frequently, however, this requirement cannot be met in practice with sufficient accuracy. As a consequence, several different values of the reflection coefficient $S_{ii}$ at any given port of the DUT are obtained from the partial 2-port measurements. At the same time, erroneous values of all the off-diagonal elements $S_{ij}$ of the $n \times n$ scattering matrix are obtained. These errors are especially large when the DUT is a well-matched, low-loss device.

The cause of these errors is the unavoidable change in the external port impedances seen by the multiport DUT from one partial 2-port measurement to another. Indeed, the various nonzero reflections from the necessarily imperfect $n-2$ auxiliary loads are frequently of the same order of magnitude as those $S_{ii}$ the device being measured would intrinsically have if terminated with perfect loads. The off-diagonal elements $S_{ij}$ of the measured $n \times n$ scattering matrix are also significantly affected by the vectorial additions of multiple reflections bouncing back and forth among the $n-2$ imperfect auxiliary loads. These mismatch-induced errors may be eliminated by rigorously taking into account the finite reflections from the $n-2$ mismatched auxiliary loads.

An exact solution of this practical problem is provided by the scattering parameter renormalization transforms originally derived, in algebraic form, by D. Woods for devices with up to six ports [2]–[4]. Unfortunately, the complex algebraic structure of these scalar transformations makes their practical application exceedingly cumbersome when the number of ports is greater than two. The recently derived matrix form of these transformations [5], however, involves only simple matrix operations, is easily programmable, and holds for any number of ports $n$. This matrix transformation is the key to calibrated measurements of the scattering matrix of multiport devices with any number of ports $n$ using a 2-port ANA or, in general, an $m$-port ANA with $m < n$. The transformation is given by [5]

$$S' = (I-S)^{-1}(S-\Gamma)(I-S\cdot\Gamma)^{-1}(I-S) \qquad (1)$$

where $S$ is the original scattering matrix normalized to a given set of port impedances $\zeta_i$ ($i=1$ to $n$), $S'$ is the transformed scattering matrix normalized to a new set of port impedences $Z_i$ ($i=1$ to $n$), $I$ is the $n \times n$ identity matrix, and $\Gamma$ is an $n \times n$ diagonal matrix containing the reflection coefficients $\Gamma_i$ ($i=1$ to $n$) of the loads $Z_i$ as seen from lines of characteristic impedance $\zeta_i$.

## II. Measurement Technique

We shall describe our measurement technique and present some of the measured results which were obtained on a commercial 3-dB, 90° hybrid coupler, using a conventional 2-port ANA. The assumed port enumeration for the coupler is shown in Fig. 1.

As explained below, the measurement of an $n$-port device using our technique requires $n$ separate auxiliary loads. However, the validity of the generalized scattering matrix renormalization algorithm does not depend on the specific magnitudes of the reflection coefficients of these loads as long as none of the loads is perfectly reflecting ($|\Gamma| = 1$) [5]. It appears, however, that for an $m$-port ANA, as many as $m-1$ of the auxiliary loads may be perfectly reflecting. To test this rigorous generality we selected an extreme

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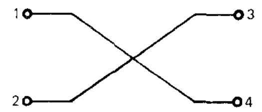


Fig 1. Port enumeration of a typical coupler.

situation that vividly demonstrates the remarkable capabilities of this new technique. The loads associated with each port of the DUT were nominally selected as: a 2:1 mismatch on port 1 ($Z_1$); a 4:1 mismatch on port 2 ($Z_2$); a 1.3:1 mismatch on port 3 ($Z_3$); and a short on port 4 ($Z_4$).

Six partial 2-port scattering parameter measurements were then performed on the 3-dB coupler with the ANA connected in turn to ports 1–2, 1–3, 1–4, 2–3, 2–4, and 3–4. In general, the number of measurements required, using a 2-port ANA, is $\binom{n}{2} = n(n-1)/2$[1] where $n$ is the number of ports. (We have implicitly assumed that the ANA used in these measurements has a test set that is capable of determining both the forward and reverse $S$-parameters in one single measurement without physically reversing the DUT.) During each 2-port measurement, the unused ports of the DUT were terminated with their corresponding loads. For example, in the first 2-port measurement, ports 1 and 2 of the DUT were connected to the ANA, port 3 was terminated with $Z_3$, and port 4 was terminated with $Z_4$. In the second 2-port measurement, ports 1 and 3 of the DUT were connected to the ANA, port 2 was terminated with $Z_2$, and port 4 was terminated with $Z_4$. Continuing this process with the remaining 2-port measurements, any unused port of the DUT, say port $i$, must always be terminated with its corresponding load $Z_i$. It is because of this rule that the measurement of an $n$-port device requires $n$ separate auxiliary loads, all fully characterized in terms of their complex reflection coefficients $\Gamma_i$.

The complex values $\Gamma_i$ of the reflection coefficients of the $n$ auxiliary loads (Figs. 2 and 3) and those $S_{ij}$ of the partial 2-port DUT measurements must all be obtained, through vector error correction, from a calibrated system. In our measurements, raw, uncalibrated data of these complex quantities were acquired and also of the scattering matrices of three calibration standards, a "thru", a "short", and a "delay". These last calibration data were then used to perform a vector error correction of all the uncalibrated scattering parameter data using the well-known TSD calibration procedure described in [7]–[11]. Alternatively, a calibration procedure based on a different set of standards could have been used instead.

After the initial TSD calibration, the six partial 2-port measurements yield six different 2×2 partial scattering matrices of the coupler, where the two ports connected to the ANA in each measurement are normalized by TSD to 50 Ω (real) while the remaining ports are normalized to the impedances of the corresponding auxiliary loads. When properly renormalized, these six 2×2 complex matrices

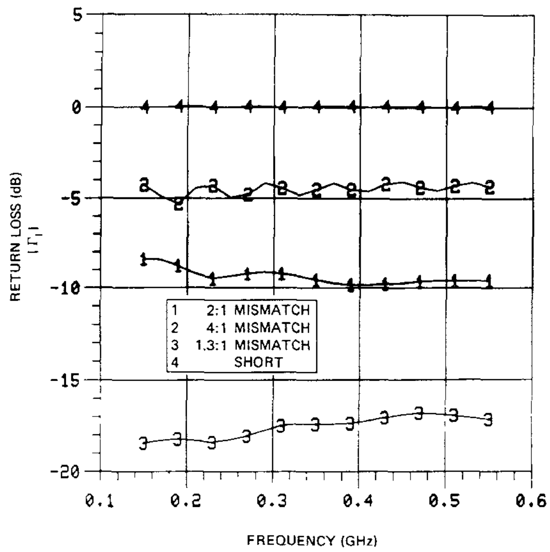


Fig. 2. Return loss of loads $Z_1$, $Z_2$, $Z_3$, and $Z_4$.

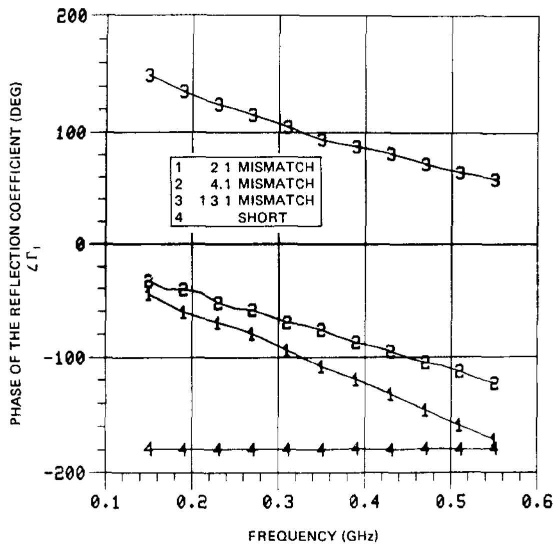


Fig 3. Reflection coefficient phase of loads $Z_1$, $Z_2$, $Z_3$, and $Z_4$.

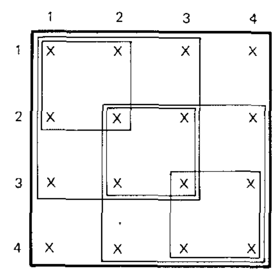


Fig. 4 The submatrices of a 4-port network

(shown schematically in Fig. 4) become submatrices of a complete and intrinsically consistent 4×4 complex scattering matrix of the 4-port coupler.

In Fig. 4 each 2×2 submatrix is represented by a square with corners at the locations of the four submatrix elements in the complete 4×4 matrix. As can be seen, the corners of three of the six squares representing the 2×2

[1]It is conjectured that for an $m$-port ANA where $m$ is even and $n$ is a multiple of $m/2$, the required number of measurements is given by $n(2n-m)/m^2$ [6]

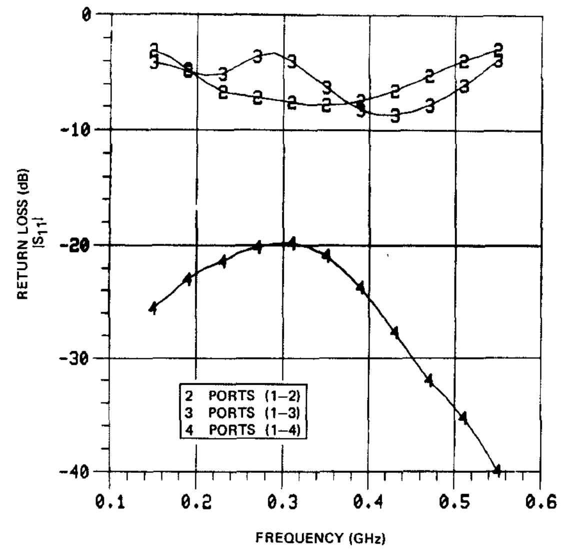


Fig. 5. Return loss at port 1 before partial renormalization.

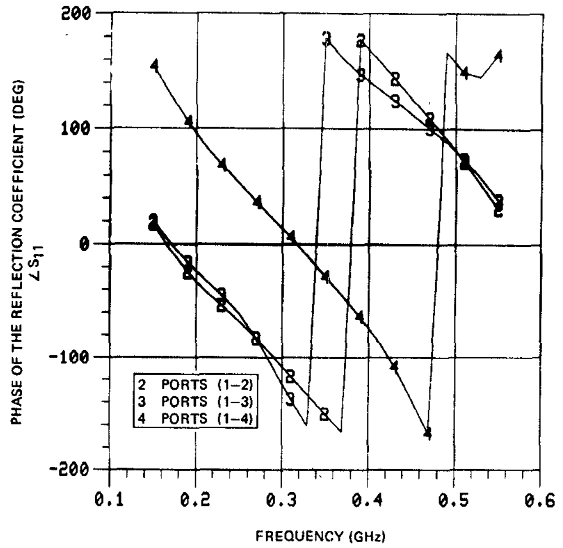


Fig. 7. Reflection coefficient phase at port 1 before partial renormalization.

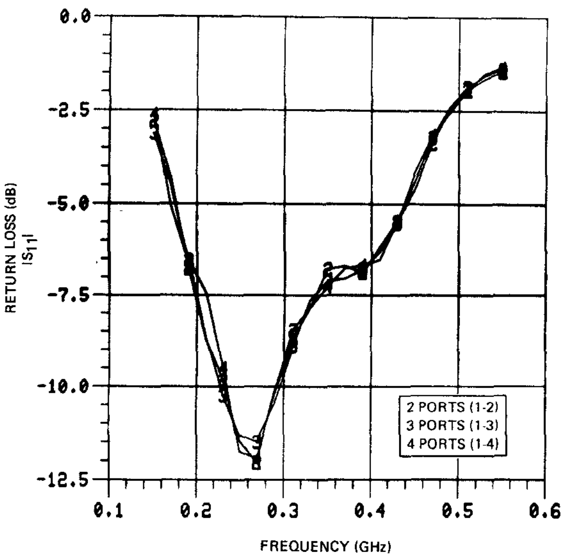


Fig. 6. Return loss at port 1 after partial renormalization.

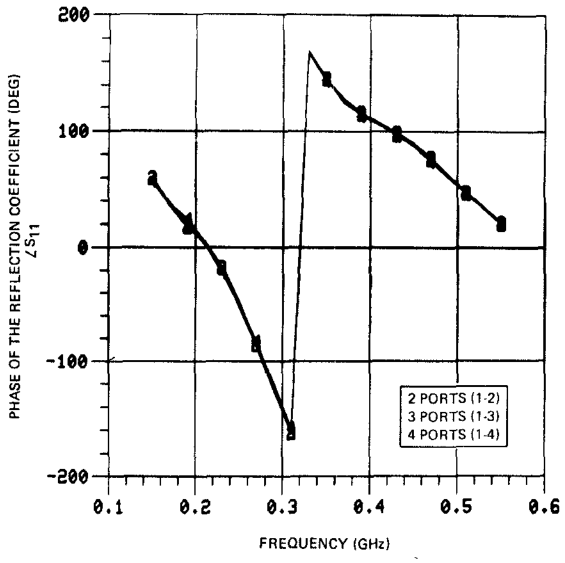


Fig. 8. Reflection coefficient phase at port 1 after partial renormalization.

submatrices are located next to each element on the diagonal of the complete 4×4 matrix. Thus, necessarily, three independent determinations are made of the reflection coefficient $S_{ii}$ at each of the four ports of the coupler. If all the auxiliary loads were perfect, the three $S_{ii}$ values obtained for each location on the diagonal would be equal to one another, and would provide a single value for the reflection coefficient at any given port. In practice, however, the three $S_{ii}$ values obtained are not mutually equal, since the external port impedances seen by the DUT in each of the partial 2-port measurements are different.

Indeed, the submatrix obtained in the first measurement (ports 1–2) is a submatrix of a complete 4×4 scattering matrix whose ports are normalized to 50, 50, $Z_3$, and $Z_4$, respectively, while the submatrix obtained in the last measurement (ports 3–4) is a submatrix of a 4×4 scattering matrix whose ports are normalized to $Z_1$, $Z_2$, 50, and 50. These mutually inconsistent port normalizations can be made consistent by renormalizing each 2×2 submatrix to the impedances of the two loads corresponding to the ports that were connected to the ANA during the partial 2-port measurement. Indeed, we renormalize the first 2×2 submatrix (ports 1–2) from 50, 50 to $Z_1$, $Z_2$, the second from 50, 50 to $Z_1$, $Z_3$, etc. After this partial renormalization is performed, each submatrix may be directly copied to the corresponding locations of a complete 4×4 scattering matrix normalized to $Z_1$, $Z_2$, $Z_3$, $Z_4$. The resulting 4×4 matrix is a first, intrinsically consistent scattering matrix of the coupler. It may be further renormalized to 50, 50, 50, 50 to reduce it to a standard form.

## III. Results

The results of the renormalization of three of the six 2×2 submatrices are shown in Figs. 5–8. Shown in Fig. 5 is the return loss at port 1 $|S_{11}|$ for the first three measurements (ports 1–2, 1–3, 1–4) before renormalization. As can

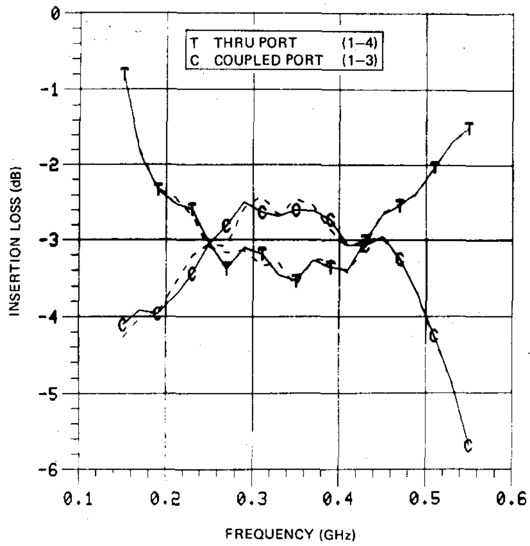


Fig. 9. Reconstructed insertion loss of a 3-dB 90° hybrid coupler.

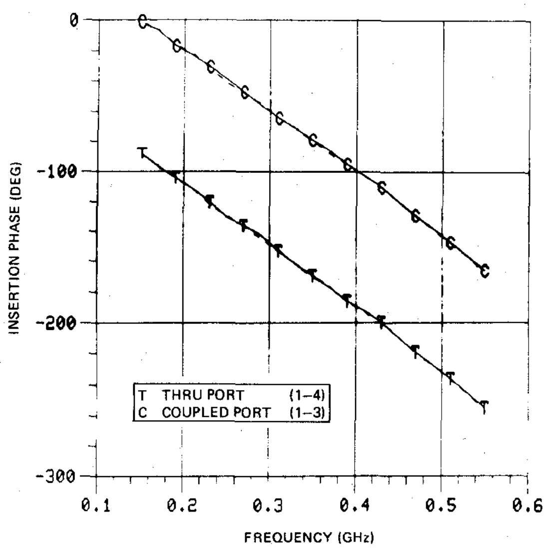


Fig. 10. Reconstructed insertion phase of a 3-dB 90° hybrid coupler.

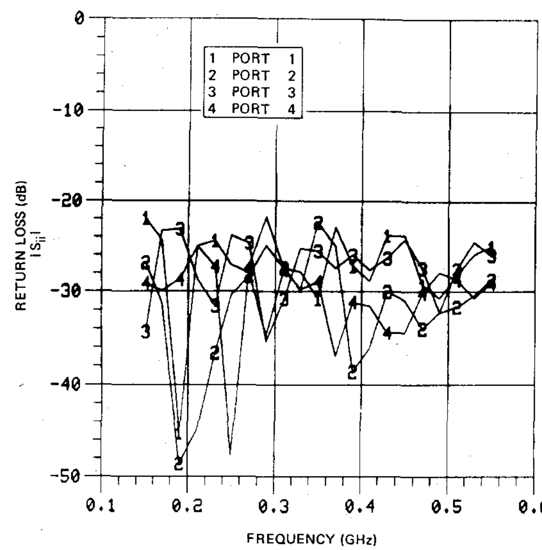


Fig. 12. Reconstructed return loss of a 3-dB 90° hybrid coupler.

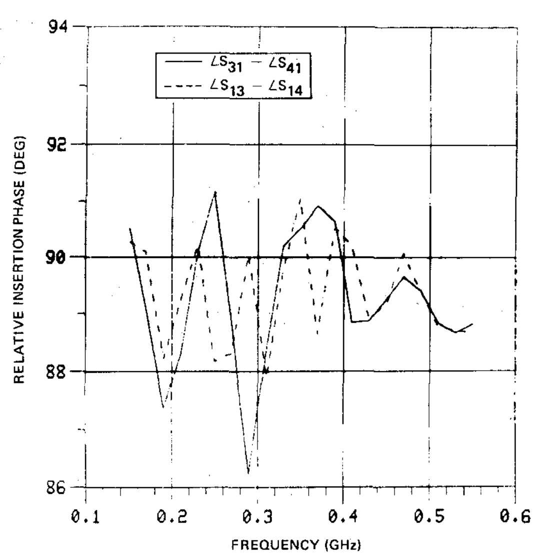


Fig. 11. Reconstructed relative insertion phase of a 3-dB 90° hybrid coupler.

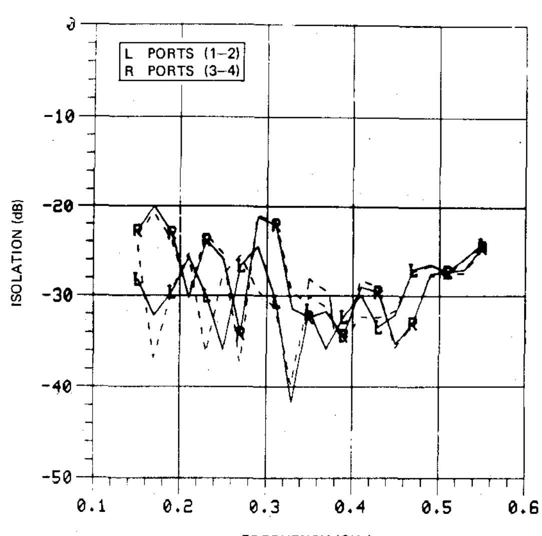


Fig. 13. Reconstructed isolation of a 3-dB 90° hybrid coupler.

be seen, the curves are quite different. After renormalization, however, the three curves become essentially identical to one another, as shown in Fig. 6. Similarly, the phase of the reflection coefficient at port 1 $\angle S_{11}$ for the first three measurements (ports 1–2, 1–3, 1–4) before renormalization is shown in Fig. 7. After renormalization, the three curves merge into one as shown in Fig. 8. Thus, the $2\times 2$ partial renormalizations generate, to within measurement accuracy, identical values for the complex reflection coefficient $S_{11}$ at port 1. Although not explicitly shown, unique values for the complex reflection coefficients $S_{ii}$ at the other ports of the coupler are also obtained. At the same time, correct values are obtained for all of the off-diagonal elements $S_{ij}$.

Once this complete $4\times 4$ scattering matrix of the coupler normalized to $Z_1$, $Z_2$, $Z_3$, and $Z_4$ has been determined,[2] it may be renormalized back to the standard set of external port impedances of 50, 50, 50, and 50. The familiar scattering matrix of a 3-dB coupler is then totally recovered, thus removing the effects of the mismatched auxiliary loads used during the partial 2-port measurements. After this final 4-port renormalization was performed, the reconstructed responses shown in Figs. 9–13 were obtained.

The insertion loss and insertion phase for the thru (lines labeled "$T$", representing $S_{14}$, $S_{41}$, $S_{23}$, and $S_{32}$) and coupled ports (lines labeled "$C$", representing $S_{13}$, $S_{31}$, $S_{24}$, and $S_{42}$) are shown in Figs. 9 and 10, respectively. In these two figures the solid lines indicate forward $S$-parameters ($S_{31}$, $S_{42}$, $S_{41}$, and $S_{32}$), and the dashed lines indicate reverse $S$-parameters ($S_{13}$, $S_{24}$, $S_{14}$, and $S_{23}$). The relative insertion phase between the thru and coupled ports is shown in Fig. 11. The nearly nominal 90° phase shift is an indication of the symmetry of the coupler. The return loss at each port of the coupler $|S_{ii}|$ is shown in Fig. 12. Finally, the isolation between ports 1 and 2 (lines labeled "$L$") and between ports 3 and 4 (lines labeled "$R$") are shown in Fig. 13.

## IV. Conclusions

At relatively low microwave frequencies where well-matched loads are readily available, this technique provides only increased measurement accuracy by eliminating relatively minor errors due to the small mismatches of the auxiliary loads. At higher frequencies, however, it becomes increasingly difficult to manufacture well-matched auxiliary loads. These would not be required, however, if this new technique were applied. In fact, even fully reflecting shorts could be used provided their total number is less than the limit indicated previously. This would seem to be an advantage, in practice, since a short is easier to manufacture and is more repeatable than most practical loads. Furthermore, a measurement of its reflection coefficient may not even be necessary. Indeed, for a good short, an assumed value of $-1$ is probably more accurate than the actual measured value one might obtain from a typical ANA.

Finally, in the case of well-matched, low-loss devices where the reflections are so small that they are difficult to measure, it may be desirable to use auxiliary loads with substantial reflection coefficients in order to make the signals reflected by the multiport DUT larger than the system's noise, and thus easier to measure accurately. In these conditions one would rely more heavily on the rigorous validity of the renormalization transform than upon the intrinsic accuracy of the auxiliary loads.

In the remaining phase of our development, an error analysis will be performed to determine the sensitivity of the reconstructed $n\times n$ scattering matrix of an $n$-port device to possible residual errors in the required values of the complex reflection coefficients of the auxiliary loads. Attempts will also be made at determining 1) optimum values of these reflection coefficients that maximize the accuracy of the reconstruction, and 2) the dependence of these optimum values on the particular response of the multiport DUT.

[2]This scattering matrix is, in general, normalized to a complex set of external port impedances. As a consequence 1) it may substantially differ from the conventional scattering matrix normalized to 50 Ω (real) at all four ports; and 2) it does not display the well-known symmetry and unitary character for reciprocal and lossless networks.

## V. Appendix

In this Appendix, we will give a brief derivation of the generalized scattering parameter transformation given by (1), as reference [5] may not be available to all interested readers.

The $Z$-matrix of an $n$-port network may be written as a function of either of the two scattering matrices $S$ or $S'$, where $S$ is normalized to a given set of port impedances $\zeta_i$ ($i=1$ to $n$), and $S'$ is normalized to a different set of port impedances $Z_i$ ($i=1$ to $n$)

$$Z=(I-S)^{-1}(I+S)\cdot\zeta_0 \qquad (2)$$

$$Z=(I-S')^{-1}(I+S')\cdot Z_0 \qquad (3)$$

where $Z_0$ and $\zeta_0$ are diagonal matrices containing the port impedances $Z_i$ ($i=1$ to $n$) and $\zeta_i$ ($i=1$ to $n$), respectively. By equating (2) and (3), one can solve for $S'$ in terms of $S$ as

$$S'=(I-S)^{-1}\left[(I+S)-(I-S)\cdot Z_0\zeta_0^{-1}\right] \cdot\left[(I+S)+(I-S)\cdot Z_0\zeta_0^{-1}\right]^{-1}(I-S). \qquad (4)$$

Finally, since

$$Z_0\zeta_0^{-1}=(I+\Gamma)(I-\Gamma)^{-1} \qquad (5)$$

upon substituting (5) into (4) one finds the result expressed by (1)

$$S'=(I-S)^{-1}(S-\Gamma)(I-S\cdot\Gamma)^{-1}(I-S). \qquad (6)$$

A completely equivalent form of the previous expression may be derived as shown in [5] with the following result:

$$S'=(I-\Gamma)^{-1}(S-\Gamma)(I-\Gamma\cdot S)^{-1}(I-\Gamma). \qquad (7)$$

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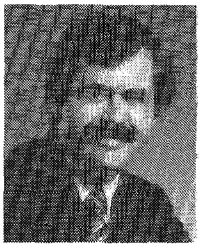

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From 1955 to 1958 he was involved in the development of microwave radio relay links, and until 1962 in the development of instrumentation for nuclear-radiation measurements and pulse-height analyzers. In 1962 he joined Philips Gloeilampen Fabrieken N. V. in Eindhoven, the Netherlands, where he served until 1965 in the Isochronous Cyclotron Development Department, and until 1970 in oscilloscope development. In late 1970 he moved to the United States to join Tektronix, Inc., Beaverton, OR, where he was involved in the development of high-speed subnanosecond circuitry for real-time oscilloscopes. In 1977 he joined TRW Defense and Space Systems Group, Redondo Beach, CA, where he is presently involved in the development of automated microwave measurement systems. He is the originator of various new methods for the computer-aided design of microwave networks and for automated measurements of scattering parameters. Among these are the well-known TSD and Super-TSD calibration methods and the new concepts of digital vector-voltmeter and multiport network analyzer.

In recently delivered theoretical contributions to the field of advanced scattering analysis of microwave networks, Dr. Speciale has introduced a generalized scattering matrix renormalization transform for multiport networks and a projective matrix transformation that describes the mapping of multiport loads in linear embedding.